SUBSTANTIATION OF MULTI-PRODUCT PRICES OPTIMIZATION MODEL

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Abstract

The paper investigates and analyses the technologies and the methods for optimization in a multi-product environment, where the firms face more difficulties to make optimal decisions. It is shown that One-Product Prices Optimization (OPPO) can't be used for Multi-Product Prices Optimization (MPPO) problem, because of both prices and demands, being random variables, may exhibit different covariate relationships, determined by means of substitution and complimentarity effects. The system of covariate relationships, determined by means of substitution and complimentarity processes, is represented as the system of multivariance regression equations. MPPO problem have been reduced to the maximization problem of the objective function of Total Revenue, which is a quadratic form. Seemingly unrelated regression models were suggested as basic methodology of estimation of MPPO model. It was shown that: if the matrix of the regressors of the model is identical for all of the regression equations (for all dependent variables), the Generalized Least Squares (GLS) method is reduced to classical Least Squares (LS) method and one can estimate regressions equations independently. Such estimations and all inferences made based on them are correct. The represented theory can be used as a basis for development of a relevant software.

Keywords: Price Optimization, Single-Product Pricing, Multi-Product Pricing, Substitution, Complementarity, Total Revenue, Generalized Least Squares, Multivariate Regression, Seemingly -unrelated Regression models.

1. Introduction and the Problem Statement

Correct and timely estimation of economic, financial and managerial parameters of an enterprise, such as: maximum of total revenue and profit, price and cross elasticity of demand, marginal revenues of the complex of products under condition of cross-elasticity correlations is one of the most demanded problems in theory and practice of economics and business.

In contrast to a single-product pricing models, multi-product pricing models have been significantly less studied due to the complexity of multi-product demand functions (Soon, 2009, pp. 399-430). Many decision-makers use incomplete demand functions which are defined only on a limited scale, e.g. the combination where all elements of demand functions are non-negative. In real marketplace managers often make pricing decisions for several products simultaneously. By doing so, decision makers can control for substitution effects or benefit from potential synergies between the products (Goic, 2011).

Companies sell several products, and in addition to determining the quantity of orders should also determine the selling price of each product sold, whereas the main component of determining the price - the demand, for each product is of conditional nature and depends on the demand of other products.

Based on the defined problem in the previous paragraph, our objectives and research goals are to create mathematical model and corresponding software that can be applied in particular pricing problems. The price optimization methods for one product are well known, but they cannot be used in the multi-product case, because of an effect of a cross-elasticity interaction among the products (Gallego, 2014, pp. 450-461) . The latter leads to complicate mutual correlations among products' prices and demands, and therefore requires usage of alternative mathematical methods. Developing methods and corresponding algorithms for solution of the defined problem allows, based on the accumulated statistical data of the prices and demands of the products, obtaining comprehensive analysis of economic, financial and managerial states of the enterprise. Specifically, the results of the research allow estimating: maximum of total revenue and profit, price and cross elasticity of demand, marginal revenues of the complex of products, break even points. It should be mentioned that there is no Business Plan without the analysis of revenue, profit, break even points; therefore, the research goal can be crucially important for easing the entire business processes.

2. Theoretical Foundations of Multi-Product Prices Optimization Models

2.1. One-Product Prices Optimization (OPPO) Problem

The OPPO problem is well known. It is based on the assumption that the dependence between Demand, D, and Price, P follows straight line

$D=a+kP,$ (1)

where k-slope, and, it has the same dimension as D, that is, number of units sold per unit of time;

a-intercept (a>0), again the same dimension as D, but let us note that it has no direct economic sense;

ε-normally distributed random variable with zero mean.

One of the basic lows of economics reveals that $k<0$, so that the increasing of Price implies strictly decreasing of Demand (Goldberger, 1991). It is clear, that this is an idealization of real economic processes (Greene, 2000), but it is a direct consequence of the assumption (1), which also implies that Total Revenue can be represented as

$TR=aP+kP^2.$ (2)

It is easy to obtain values of prices and demand which provide maximum of Total Revenue. To find maximum of the function (2) one has to differentiate it:

$\frac{dTR}{dp}= a+2kP$

making it equal to zero gives simple equation

$a+2kP=0,$

which leads to optimal value of the price variable, that is to the price value which maximizes Total revenue function.

$P_{op}=\frac{a}{2k};$ (3)

Substituting the later into (1) implies corresponding value of optimal Demand

$D_{op}=\frac{a}{2};$ (4)

The production of (3) and (4) defines the maximum value of the Total Revenue function

$Tr_{max} = \frac{a^2}{4k}.$ (5)

The next step is including the costs in the model.

The cost function can be represented by means of straight line equation

$C= C_F+c_vD,$ (6)

where C_F-fixed costs, and C_v – variable costs.

Including of the costs requires inverting of (1)

$P=a_1- k_1D$, (1')

where $a_1=a/k$; $k_1=1/k$.

Now the Total Revenue function can be expressed in terms of Demand,

$TR=a_1D+k_1D^2$,

and one can use (6) to calculate Profit function and points D_1 and D_2 where Profit is equal to costs, that is Break-even points.

$Profit=TR-C= a_1D-k_1D^2- C_F-c_vD=0$;

The latter is a simple quadratic equation (with respect to unknown D), which can be easily solved

$$D_{1,2} = \frac{-(c_v-a_1)\pm\sqrt{(c_v-a_1)^2+4*C_F*k_1}}{2k_1}. \quad (7)$$

Also, it is easy to calculate parameters corresponding to (3)÷(5)

$$P_{op}^{c} = \frac{a_1+c_v}{2}; \quad (3')$$

$$D_{op}^{c} = \frac{a_1-c_v}{2k_1}; \quad (4')$$

$$Prmax = \frac{a_1^2-c_v^2}{4k_1}. \quad (5')$$

All of these parameters are extremely useful for implementation of efficient financial and economic management in a retail business. Note, that they are strongly depended on correct and reliable estimation of regression parameters a, k and ε in (1).

2.2. Development of the Mathematical Model for the MPPO problem

Despite of the usefulness of the model, it cannot be applied when a manager faces Multi-Product Prices Optimization (MPPO) problem. Beside that the latter Problem differs from OPPO problem with high dimensions of the all functions represented above, it has another specific issue: both prices and demands may exhibit different covariate relationships, determined by means of substitute and compliment processes. Such kind of covariate relationships create a system of constraints which should be included in the relevant mathematical model of the MPPO problem.

2.2.1. Determination of the model

The basic objects of the MPPO model are observed random variables: d_i (i=1, 2,…,m) - demand and p_i (i=1,2,…,m)-corresponding prices. We don't consider question about their distributions. They constitute two m-dimensional vectors d (d_1,…,d_m) and p(p_1,…, p_m). We assume that there exist m relationships of type (1) between their components

$d_i=a_i+k_{ii}p_i +\varepsilon_i$ (i=1, 2,…,m), (8)

where all parameters have the same sense as in (1). We call (8) paired models and their parameters - paired parameters. Note that paired model is nothing but the set of m independent OPPO models.

Total Revenue function TR_M of the MPPO problem can be represented as an inner product of two m-dimensional vectors d and p

$$TR_M = (d, p), \quad (9)$$

or, in coordinate form

$$TR_M = \sum_{i=1}^{m} d_i p_i. \qquad (10)$$

Drastic distinction between paired models and MPPO models is that each of the demand potentially can be dependent on all/part of price variables

$$\begin{aligned} p_1 &= \gamma_{11} d_1 + \ldots + \gamma_{1m} d_m + b_1 + \varepsilon_1 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots , \\ p_m &= \gamma_{m1} d_1 + \ldots + \gamma_{mm} d_m + b_m + \varepsilon_m \end{aligned} \qquad (11)$$

Where p_i – price of the item i;

d_i - demand of the item i;

ε_i - normally distributed random variable with zero mean corresponding to observations of item i.

Observe, that γ_{ij} (i, j=1,2,…,m) now may have positive or negative signs, as demands can be in different relations (substitute or compliment) with different prices.

(11) can be represented in more economic, matrix, format

$$p = \Gamma p + b + \varepsilon, \qquad (12)$$

where d and p - m×1 vectors of demand and prices;

b- m×1 vector of intercepts;

Γ – m-order matrix, with entries γ_{ij} equal to coefficients in (11) and subject to identification.

ε - random m×1 vector, contained normally distributed random variables with zero mean and covariance matrix Σ, which completely defines statistical nature of the system of equations in (12), and therefore their identification method. This question will be discussed latter in detail.

If the entries γ_{ij} of the matrix Γ have already being identified, then substituting of (12) into (10) implies (after simple transformations)

$$TR_M = p^T K p + (p, b), \qquad (13)$$

where $K -$ simmetric matrix with entries $k_{ij} = \begin{cases} \gamma_{ij} & if\ i = j \\ \frac{(\gamma_{ij}+\gamma_{ji})}{2} & if\ i \neq j \end{cases}$ (i=1, 2,…,m).

Considering natural constraints of non-negativity of prices,

$$0 \le p_i \text{ (i=1, 2,…,m)} \qquad (14)$$

the MPPO problem have been reduced to the maximization problem of the objective function (13) (which is a quadratic form with symmetric matrix K) with constraint (14). Thus, the MPPO problem can be solved by means of well-known quadratic programming methodology.

2.2.2. General definition of the Quadratic programming

The general quadratic programming problem (GQPP) can be written as minimize

$$f(\mathbf{x}) = \mathbf{cx} + \frac{1}{2}\mathbf{x}^{T}\mathbf{Q}\,\mathbf{x}, \qquad (15)$$

subject to

$$\mathbf{Ax} \le \mathbf{b} \text{ and } \mathbf{x} \ge \mathbf{0}, \qquad (16)$$

where

c - is an n-dimensional row vector describing the coefficients of the linear terms in the objective function;

Q - is an (n □□n) matrix of the coefficients of the quadratic terms. Important to note that the matrix is symmetric;

x – n-dimensional column vector of the decision variables;

A – is a (m □□n) matrix of the constraints;

b - m-dimensional column vector of right-hand-side coefficients.

Solution of the above formulated problem is well known Karush-Kuhn-Tucker method. The method is based on Lagrange multipliers approach. The Lagrangian function for the quadratic program is

$L(\mathbf{x}, \boldsymbol{\mu}) = \mathbf{cx} + (1/2)\ \mathbf{xQ}\ \mathbf{x} + (\mathbf{Ax} - \mathbf{b})$,

where $\boldsymbol{\mu}$ is an m-dimensional row vector. The Karush-Kuhn-Tucker conditions for a local minimum are given as follows.

$$\frac{\partial L}{\partial x_j} \geq 0, j = 1, \cdots, n$$

$$c + x^T Q + \mu A \geq 0$$

$$\frac{\partial L}{\partial \mu_i} \leq 0, i = 1, \cdots, m$$

$$Ax - b \leq 0$$

$$x_j \frac{\partial L}{\partial x_j} = 0, j = 1, \cdots, n$$

$$x^T(c^T + Qx + A^T \mu) = 0$$

$$\mu_i g_i(x) = 0, i = 1, \cdots, m$$

$$\mu(Ax - b) = 0$$

$$x_j \geq 0, j = 1, \cdots, n$$

$$x \geq 0$$

$$u_i \geq 0, i = 1, \cdots, m$$

$$\mu \geq 0$$

We will not discuss the GQPP for some reasons, which will be clarified below, but only note that when the objective function f(x) is strictly convex the problem has a unique local minimum which is also the global minimum (the problem of local maximum can be easily converted into the problem of local minimum by means of assigning opposite sign to objective function (15)).

2.2.3. Existence of MPPO problem.

It is clear, that we must detect conditions of existence of maximum of defined MPPO problem. Comparing the MPPO model (13), (14) with the GQPP model (15), (16) permits to make the following conclusion: MPPO model is, from mathematical point of view, simpler than GQPP because of the simplest constraints (14), which represents non-negativeness of variables. The latter means that one has to detect existence of global maximum of the MPPO model. The maximum, if it exists, will be placed within the area of positive values of variables. It implies, that we do not need to analyze constraints similar to constraints (16), design Lagrange function and investigate it. Our task can be now formulated as calculating global maximum of quadratic form (13).

Quadratic forms have one critical point, which may be one out of three types: maximum, minimum or saddle points. Type of the points depends on signs of entries of the matrix K. In one dimensional case we have guaranteed maximum, because of negative sign of slope k and positive sign of intercept a in (1), which are consequences of economic lows. Situation is different in multi-dimensional case, where entries of the matrix Γ, and therefore of the matrix K, may have different signs, which is determined by complicate substitute-compliment correlative relationships. It means that in multidimensional cases we faced the problem of existence of maximum of Total Revenue function, and that, in certain cases, the maximum may not exist at all.

Now we should clarify a criterion to detect whether the MPPO problem has the maximum point in the certain case. Such criterion can be constructed on the base of the Hessian matrix, which is the matrix of the second derivatives of TR_M quadratic forms with respect to prices.

Let's denote the Hessian matrix as H, whereas the matrix of the first derivatives as - $K^{(1)}$. The m×1 vector of the first derivative of quadratic form (13) will be

$$TR_M^{(1)} = K^{(1)}p + b \qquad (17)$$

where the entries of the matrix $K^{(1)}$ are

$$k_{ij}^{(1)} = \begin{cases} 2\gamma_{ij} & if\ i = j \\ \gamma_{ij} + \gamma_{ji} & if\ i \neq j \end{cases} \text{ (i,j=1, 2,…,m).}$$

It follows from (15) that the Hessian coincides with $K^{(1)}$.

It is well known fact that if the matrix H is positively definite at the critical point then quadratic form (13) has strict minimum, if the matrix is negatively definite then quadratic form (13) has strict maximum. The definitions are general, and directly cannot be used as a criterion to detect whether there is a maximum. With this in view the following property of Hessian matrix can be used. Because the Hessian matrix is symmetric matrix its eigenvalues are real numbers, which implies the following important properties of Hessian.

Suppose x_0 is a critical point for a function f and λ_i are the eigenvalues of the Hessian matrix. Then:

(a) If all of the eigenvalues $\lambda_i > 0$, (i=1, 2,…,m) then at point x_0 is a strict local minimum of function f;

(b) If all the eigenvalues, $\lambda_i < 0$ (i=1, 2,...,m) then at point x_0 is a strict local maximum of function f.

(c) If at least one eigenvalue is positive and at least one eigenvalue is negative then at point $x_0$is a saddle point of function f.

The property (b) can be used as reliable and simple criterion of existing the optimal solution of the MPPO problem. The criterion can be reformulated in our case: if all the eigenvalues of the Hessian of the objective quadratic form of Total Revenue, $\lambda_i<0$ (i=1, 2,...,m) then the problem has strict maximum.

We have already mentioned above that the maximum of the MPPO problem may not be existed at all, and the reason of this is the complicate substitute-compliment correlative relationships. Thus we can confirm that for the multidimensional case absence of local maximum of TR_M function of the MPPO is internal property of Multi-Product situation, which is consequence of economic laws.

2.3. Identification problem of the parameters of the model of MPPO problem.

Besides the question connected with the negative definiteness of the matrix K (Hessian of the quadratic form (13)), there exists another problem: identification of all entries of the matrix K. It is clear that without reliable estimation these values on the base of existed observations cannot speak about solution of optimization problem.

2.3.1. The seemingly unrelated regressions (SUR) model

The question arises: how one should estimate entries of the matrix Γ in the system (12). The point is that there exist two approaches [1],[2] .

The first one is estimating γ_{ij} (i, j=1,2,…,m) on the base of each equation separately, assuming that the errors ε_i (i=1,2,…,m) are distributed independently, i.e. there are no cross-correlations between them (ε_i, ε_j)=0 (i, j=1,2,…,m)[1] [2].

The second one is estimating γ_{ij} (i, j=1,2,…,m) on the base of assumption that (ε_i, ε_j)≠0 (i, j=1,2,…,m), i.e. assuming that random variables of error ε_i (i=1,2,…,m) are not independent. The latter leads to some specific estimative problems. The especial class of models- the seemingly unrelated regressions (SUR) model-were elaborated for such cases [1]. Below we gave short review and analysis of SUR models, because this technology has decisive impact on our research [1],[3]

We start with mentioning that SUR models are case of more general Generalized Least Squares (GLS) models (Zellner, 1962).

In this section we use the general notations for multivariate regression analysis:

M-number of dependent variables y_i (i=1, 2,…,M);

N- number of observations for each i.

It means that we consider M different regressions for equal number of observations. Also we consider K_i independent variables X_{ij} (j=1, 2,…, K_i). To clarify the latter notation consider, for example, the variable $X_{35.}$ It represents the fifth variable in regression number 3 (the fifth variable for the third dependent variable y_3). Thus, regression i has K_i independent variable $X_{i.}$ Taking into account that there are N number of observations for each I, one can say that we have M matrices of independent factors of sizes N×K_i (i=1, 2,…,M) each.

Also, one can express it more compact way (Baltagi, 2008)in terms of matrix notation:

$$y_j = X_j \beta_j + \varepsilon_j, \qquad (18)$$

where
j = 1, ..., M;
y_j and ε_j are N-vectors and X_j is an N × K_j matrix:
K_j = dim(β_j) - the number of regressors for the jth regression.
The conventional assumptions of the classical regression models are assumed to hold for each regression j (j = 1, ..., M):

[1] Recall that we have assumed that errors ε_i are normal random variables, so in the case notions of independence and non-correlation are coincide.

$E(y_j) = X_j\,\beta_j,$ (19)

$V(y_j) = \sigma_{jj} I_N\,,$ (20)

Where:

E - symbol of operation of calculation of expectation of a stochastic variable;

V- symbol of operation of calculation of variance of a stochastic variable;

σ_{jj} -diagonal element covariance matrix of of a stochastic variable i;

I_N – identity matrix of rank N.

X_j is non-stochastic matrix of regressors;

rank $(X_j) = K_j$.

Note, that (20) means that classical regression assumes the dependent variables are uncorrelated. It is very important assumption, as it justifies all inferences made based on the equation (19). Additional assumption of multinormality of dependent variables y_j (j = 1, ..., M), classical LS estimator of βj can be applied separately to each equation [4],[5].

We cannot assume it in our case, as it is clear, that prices on goods have cross correlations, due to substitute and complimentarity effects. Namely, SUR model permits nonzero covariance between the error terms ε_{ij} and ε_{ik} for a given observation i across regressions j and k [1],[5]:

$Cov(\varepsilon_{ij}\,,\, \varepsilon_{ik}) = \sigma_{ij},$ (21)

but

$Cov(\varepsilon_{ij}\,,\, \varepsilon_{i'k}) = 0$ (22)

if $i \neq i'$.

If one considers prices of m goods as dependent variables, then (21) and (22) means that prices of different goods are cross-correlated, but only within simultaneous observations. The latter assumption is very interesting from business point of view.

In matrix form

$Cov(\varepsilon_j,\varepsilon_k) = \sigma_{jk} I_N.$

The nonzero covariance across equations j and k defines specific features of estimation methodology of SUR models (Generalized Least Squares (GLS)estimation) and its differences from classical Least Squares (LS) estimation (Strutz, 2016).

To use compact and easy understandable matrix notations we introduce the following notations (geometrical dimensions of the enumerated geometrical objects are shown below the objects) (Steeb, 2006)

$$\underset{(MN\times 1)}{y} = \begin{vmatrix} y_1 \\ y_2 \\ \cdots \\ y_M \end{vmatrix}$$ -a vector of M dependent variables;

$$\underset{(MN\times 1)}{\varepsilon} = \begin{vmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \cdots \\ \varepsilon_M \end{vmatrix}$$ -a vector of M vector errors;

$$\underset{(K\times 1)}{\beta} = \begin{vmatrix} \beta_1 \\ \beta_2 \\ \cdots \\ \beta_M \end{vmatrix}$$- a vector of K vectors of regression parameters to be identified

and

$$\underset{(MN\times K)}{X} = \begin{vmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & X_M \end{vmatrix}$$-block matrix of regressors,

where $K = \sum_{j=1}^{M} K_j$.

Now, it is easy to express expectation of the vector of M dependent variables as

$E(y) = X\beta$,

as it was expected.

To continue, we have to use tensorial algebra notations, specifically tensorial products (Kronecker's product) methodology [6]. According to its definition, for matrices A and B the product can be represented as

$$(A \otimes B) = \begin{vmatrix} a_{11}B & a_{12}B & \cdots & a_{1M}B \\ a_{21}B & a_{22}B & \cdots & a_{2M}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{M1}B & 0 & \cdots & a_{MM}B \end{vmatrix}. \quad (23)$$

It is not difficult to show that such definition of the product leads to the following features of the tensorial product [6]:

$$(A \otimes B)(C \otimes D) = AC \otimes BD \quad (24)$$

and

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}, \quad (25)$$

Now, one can write covariance matrix of the dependent variables as

$V(y) = \Sigma \otimes I_N$

or in more detailed form

$$\underset{(MN\times MN)}{V(y)} = \begin{vmatrix} \varepsilon_{11}I_N & \varepsilon_{12}I_N & \cdots & \varepsilon_{1M}I_N \\ \varepsilon_{21}I_N & \varepsilon_{22}I_N & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \varepsilon_{M1}I_N & 0 & \cdots & \varepsilon_{MM}I_N \end{vmatrix},$$

where

$$\underset{(M\times M)}{\Sigma} = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1M} \\ \sigma_{21} & \sigma_{22} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{M1} & \sigma_{M2} & \cdots & \sigma_{MM} \end{vmatrix}. \quad (26)$$

Using the notations introduced, the estimations for classical Least Squares (LS) can be expressed as follows

$\widehat{\beta_{LS}} = (X'X)^{-1}X'y =$

$$= \begin{vmatrix} (X'X)^{-1}X_1'y_1 \\ (X'X)^{-1}X_2'y_2 \\ \dots \\ (X'X)^{-1}X_M'y_M \end{vmatrix} \quad (27)$$

whereas the generalized one (GLS) will be

$$\widehat{\beta_{GLS}} = (X'(\Sigma \otimes I_N)^{-1}X)^{-1}X'(\Sigma \otimes I_N)^{-1}y =$$

$$= (X'(\Sigma^{-1} \otimes I_N)X)^{-1}X'(\Sigma^{-1} \otimes I_N\}y =$$

$$= \begin{vmatrix} \sigma^{11}(X_1'X_1) & \sigma^{12}(X_1'X_2) & \cdots & \sigma^{1M}(X_1'X_M) \\ \sigma^{21}(X_2'X_1) & \sigma^{22}(X_2'X_2) & \cdots & \sigma^{2M}(X_2'X_M) \\ \vdots & \vdots & \vdots & \vdots \\ \sigma^{M1}(X_M'X_1) & \sigma^{M2}(X_M'X_2) & \cdots & \sigma^{MM}(X_{MM}') \end{vmatrix} \begin{vmatrix} X_1'\left(\sum_j \sigma^{1j} y_j\right) \\ X_2'\left(\sum_j \sigma^{2j} y_j\right) \\ \dots \\ X_M'\left(\sum_j \sigma^{Mj} y_j\right) \end{vmatrix}. \quad (28)$$

Here we used properties (24) and (25) of the Kronecker's product (23).

Thus, one can see that GLS estimation methodology requires estimation of covariance matrix (26) and calculation of its inverse. The latter is not always possible, because the matrix may not have inverse, furthermore, matrix (26) can appear be ill-conditioned, so attempt to calculate its inverse implies non-stability, which is hardly overcome problem.

Fortunately, specific character of MPPO problem allows significant simplification of GLS methodology. Point is that in this problem we have the same matrix of regressors (demands) for all dependent variables (prices). It means that for our problems the matrix of regressors can be represented as

$$\underset{(MN \times K)}{X} = \begin{vmatrix} X_0 & 0 & \cdots & 0 \\ 0 & X_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & X_0 \end{vmatrix}. \quad (29)$$

Substituting the (29) matrix X into (28) implies

$$\widehat{\beta_{GLS}} = (X'(\Sigma \otimes I_N)^{-1}X)^{-1}X'(\Sigma \otimes I_N)^{-1}y =$$

$$\left((I_M \otimes X_0)'(\Sigma^{-1} \otimes I_N)(I_M \otimes X_0)\right)^{-1}(I_M \otimes X_0)'(\Sigma^{-1} \otimes I_N)y =$$

$$= (\Sigma^{-1} \otimes X_0'X_0)^{-1}(\Sigma^{-1} \otimes X_0')y$$

$$= (I_M \otimes (X_0'X_0)^{-1}X_0') \begin{vmatrix} y_1 \\ y_2 \\ \dots \\ y_M \end{vmatrix} =$$

$$= \begin{vmatrix} (X_0'X_0)^{-1}X_0'y_1 \\ (X_0'X_0)^{-1}X_0'y_2 \\ \dots \\ (X_0'X_0)^{-1}X_0'y_M \end{vmatrix} =$$

$$= \widehat{\beta_{LS}}. \quad (30)$$

Here again the properties (24) and (25) of the Kronecker's product (23) were used.

The expression (30) shows very important result: if the matrix of the regressors is the identical for all the regression equations (for all dependent variables), the GLS method is reduced to classical LS

and one can estimate regressions equations independently. Such estimations and all inferences made on their base are correct.

It allows avoiding above mentioned problems of possible ill-conditioness of the estimated covariance matrices, which simplifies the mathematical model of the MPPO problem and makes it more reliable. Note, that the covariance matrix of variables can now be estimated as

$$\hat{\sigma}_{jk} \equiv \frac{1}{N}\left(y_j - X_j\widehat{\beta_j}\right)'\left(y_k - X_k\widehat{\beta_k}\right);$$

$$\hat{\sigma}_{jk} = \begin{cases} \hat{\sigma}_{jj} & j = k \\ 0 & j \neq k \end{cases},$$

where $\widehat{\beta_j}$ − classical Least Squares (LS) estimations (30).

Going back to the notations of the regression equations based model (11) or (12) of the MPPO problem, one can rewrite the latter results as

$$\Gamma = \begin{vmatrix} (d'd)^{-1}d'p_1 \\ (d'd)^{-1}d'p_2 \\ \dots \\ (d'd)^{-1}d'p_m \end{vmatrix}, \qquad (31)$$

$\Gamma = \{\gamma_{ij}\}$ $(\mathrm{i}, \mathrm{j} = 1,2, \dots, \mathrm{m})$ - m×m matrix of estimated parameters of the system of regression equations (11).

3. Conclusion

As it was mentioned above, one can estimate regressions equations of the model (11) independently for each item of goods. It means that we have right to use multivariable regression analysis technique, and consider price of the each item of goods as a function of m demand, including proper demand. But the latter, in turn, allows to use maximization methods also independently for each estimated regression equation. Thus, we completely substantiated nature and methodology of the mathematical model of the MPPO problem.

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